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## Chromatic equivalence classes of certain generalized polygon trees

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### Abstract

Let  $P(G)$  denote the chromatic polynomial of a graph  $G$ . Two graphs  $G$  and  $H$  are chromatically equivalent, written  $G \sim H$ , if  $P(G) = P(H)$ . Let  $\mathcal{G}$  denote the family of all generalized polygon trees with three interior regions. Xu (1994) showed that  $\mathcal{G}$  is a union of chromatic equivalence classes under the equivalence relation ' $\sim$ '. In this paper, we determine infinitely many chromatic equivalence classes in  $\mathcal{G}$  under ' $\sim$ '. As a byproduct, we obtain a family of chromatically unique graphs established by Peng (1995).

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### 1. Introduction

The graphs that we consider are finite, undirected and simple. Let  $P(G)$  denote the chromatic polynomial of a graph  $G$ . Two graphs  $G$  and  $H$  are said to be *chromatically equivalent*, and we write  $G \sim H$ , if  $P(G) = P(H)$ . Trivially, the relation ' $\sim$ ' is an equivalence relation on the class of graphs. A graph  $G$  is *chromatically unique* if for any graph  $H$  satisfying  $H \sim G$ , we have  $H \cong G$ ; that is,  $\langle G \rangle = \{G\}$  (up to isomorphism), where  $\langle G \rangle$  denotes the chromatic equivalence class determined by  $G$  under ' $\sim$ '.

A path of length  $k$  is denoted by  $P_k$ , and a cycle of length  $k$  by  $C_k$ . A path in  $G$  is called a *simple* path if the degree in  $G$  of each interior vertex is two. A *generalized polygon tree* is a graph defined recursively as follows. A cycle  $C_p$ ,  $p \geq 3$ , is a generalized polygon tree. Next, suppose  $H$  is a generalized polygon tree containing a simple path  $P_k$ , where  $k \geq 1$ . If  $G$  is a graph obtained from the union of  $H$  and a cycle  $C_r$ , where  $r > k$ , by identifying  $P_k$  in  $H$  with a path of length  $k$  in  $C_r$ , then  $G$  is also a generalized polygon tree. In this paper, we shall study the chromatic equivalence of graphs in the family  $\mathcal{G}$  of generalized polygon trees with three interior regions.

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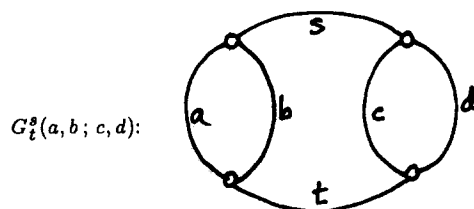


Fig. 1.

Consider the generalized polygon tree  $G_t^s(a, b; c, d)$  with three interior regions shown in Fig. 1. The integers  $a, b, c, d, s$  and  $t$  represent the lengths of the respective paths between the vertices of degree 3, where  $s \geq 0, t \geq 0$  and  $\min\{a, b, c, d\} \geq 2$ . By symmetry we can, without loss of generality, assume that  $a \geq b, c \geq d$  and  $b \geq d \geq 2$ . Thus,  $\min\{a, b, c, d\} = d$ . Let  $r = s + t$ . We now form a family  $\mathcal{C}_r(a, b; c, d)$  of the graphs  $G_t^s(a, b; c, d)$  where the values of  $a, b, c, d$  and  $r$  are fixed but the values of  $s$  and  $t$  vary; that is

$$\mathcal{C}_r(a, b; c, d) = \{ G_t^s(a, b; c, d) \mid r = s + t, s \geq 0, t \geq 0 \}.$$

For example,  $\mathcal{C}_4(6, 4; 5, 3) = \{G_4^0(6, 4; 5, 3), G_3^1(6, 4; 5, 3), G_2^2(6, 4; 5, 3)\}$  and  $\mathcal{C}_1(6, 4; 5, 3) = \{G_1^0(6, 4; 5, 3)\}$ . It is clear that the families  $\mathcal{C}_0(a, b; c, d)$  and  $\mathcal{C}_1(a, b; c, d)$  are singletons. In general, there are exactly  $(r+1)/2$  nonisomorphic graphs in the family  $\mathcal{C}_r(a, b; c, d)$ .

Xu et al. [11] showed that  $\mathcal{C}_0(a, b; c, d)$  is a chromatic equivalence class if and only if  $\{a, b, c, d\} \neq \{2, k, k+1, k+2\}$  for any integer  $k \geq 2$ . Recently, Peng [5] established that  $\mathcal{C}_1(a, b; c, d)$  is also a chromatic equivalence class, that is  $\mathcal{C}_1(a, b; c, d) = \langle G_1^0(a, b; c, d) \rangle$ , if  $\min\{a, b, c, d\} \geq 4$ . Teo and Koh [8] proved that  $\mathcal{C}_r(a, b; 2, 2)$  is a chromatic equivalence class for any integer  $r \geq 1$ . In [4], Koh and Teo conjectured that the family of graphs  $\mathcal{C}_r(a, b; c, d)$  is a chromatic equivalence class for any  $r \geq 2$ . In other words,  $\mathcal{C}_r(a, b; c, d) = \langle G_t^s(a, b; c, d) \rangle$  for any  $r \geq 2$ . However, this conjecture turns out to be false because  $\langle G_2^1(4, 2; 3, 2) \rangle = \mathcal{C}_3(4, 2; 3, 2) \cup \{G_1^0(5, 3; 3, 2)\}$ . In general, Chen and Ouyang [3] showed that  $\mathcal{C}_r(a, b; 3, 2)$  is a chromatic equivalence class if and only if  $(a, b, r) \neq (k, k+2, k+1)$  or  $(k+1, k+3, k-1)$ , for some  $k \geq 2$ . This result naturally leads one to ask what are the necessary and sufficient conditions for  $\mathcal{C}_r(a, b; c, d)$  (where  $a+b$  and  $c+d \neq 5$ ) to be a chromatic equivalence class. In this paper, we present some sufficient conditions for the family of generalized polygon trees  $\mathcal{C}_r(a, b; c, d)$  to be a chromatic equivalence class.

In the remainder of this paper, we always assume that  $r \geq 1$ . For terms used but not defined here, the reader is referred to Read [7], and Chartrand and Lesniak [1].

## 2. Basic results

In this section, we state some known results and prove some preliminary lemmas which are useful in establishing our main theorems in the next two sections.

We begin with the following result, due to Whitney [9] (see also [7]).

**Theorem 1.** *Let  $G$  be a graph and  $e$  an edge in  $G$ . Then*

$$P(G) = P(G - e) - P(G \cdot e)$$

where  $G - e$  is the graph obtained from  $G$  by deleting  $e$ , and  $G \cdot e$  is the graph obtained from  $G$  by contracting the two vertices joined by  $e$  and removing one member of each resulting pair of multiple edges.

Suppose that  $G_1$  and  $G_2$  are graphs containing a complete subgraph  $K_r$ , where  $r \geq 1$ . A graph is a  $K_r$ -gluing of  $G_1$  and  $G_2$  if it is obtained from the union of  $G_1$  and  $G_2$  by identifying a copy of  $K_r$  in each, in an arbitrary way. The following result of Zykov [12] simplifies the computation of  $P(G)$  when  $G$  is a  $K_r$ -gluing of graphs.

**Theorem 2.** *Let  $G$  be a  $K_r$ -gluing of graphs  $G_1$  and  $G_2$ . Then*

$$P(G) = \frac{P(G_1) P(G_2)}{\lambda(\lambda - 1) \cdots (\lambda - r + 1)}.$$

The following theorem lists some well-known necessary conditions for chromatic equivalence. We denote the girth of a graph  $G$  by  $g(G)$ . (The girth of  $G$  is the length of a shortest cycle of  $G$ .)

**Theorem 3.** *Let  $G$  and  $H$  be chromatically equivalent graphs. Then*

- (a)  $|V(G)| = |V(H)|$ ;
- (b)  $|E(G)| = |E(H)|$ ;
- (c)  $g(G) = g(H)$ ;
- (d)  $G$  and  $H$  have the same number of shortest cycles.

A pair  $\{u, v\}$  of nonadjacent vertices of a graph  $G$  is called an *intercourse pair* if there are at least three internally disjoint paths joining  $u$  and  $v$  in  $G$ . Let  $c(G)$  denote the number of intercourse pairs of vertices in  $G$ . We call  $c(G)$  the *intercourse number* of  $G$ . A crucial result is as follows.

**Theorem 4** (Xu [10]). *Let  $G$  and  $H$  be two chromatically equivalent graphs. If  $G$  is a generalized polygon tree, then  $H$  is also a generalized polygon tree, and  $c(G) = c(H)$ .*

Let  $H$  be a nonempty graph with two nonadjacent vertices  $u$  and  $v$ . We denote by  $H^*$  the graph obtained from  $H$  by identifying the vertices  $u$  and  $v$  of  $H$ . Let  $G_1$  (respectively,  $G_2$ ) be the graph obtained from  $H \cup P_s$  by identifying the end vertices of the path  $P_s$  with the vertices  $u$  and  $v$  (respectively, with any two adjacent vertices) of  $H$ . Then the calculation of  $P(G_1)$  can be shortened by using the following formula [6]:

$$P(G_1; \lambda) = P(G_2; \lambda) + (-1)^s P(H^*; \lambda). \quad (1)$$

We shall use the next result to prove Lemma 2.

**Theorem 5** (Xu et al. [11]). *If  $G_0^0(a', b'; c', d') \sim G_0^0(a, b; c, d)$ , then  $\{a', b', c', d'\} = \{a, b, c, d\}$ .*

In the remainder of this section, we shall prove two preliminary lemmas which are important for us in establishing many chromatic equivalence classes in  $\mathcal{G}$ .

**Lemma 1.** *All the graphs in  $\mathcal{G}_r(a, b; c, d)$  are chromatically equivalent.*

**Proof.** Let  $G_t^s(a, b; c, d)$  and  $G_{t'}^{s'}(a, b; c, d)$  be any two graphs in  $\mathcal{G}_r(a, b; c, d)$  where  $r = s + t = s' + t'$ . Then by using Theorem 1 it is a straightforward exercise to show that  $P(G_t^s(a, b; c, d)) = P(G_{t'}^{s'}(a, b; c, d))$ .  $\square$

**Lemma 2.** *If  $G_{t'}^{s'}(a', b'; c', d') \sim G_t^s(a, b; c, d)$  and  $\{a' + b', c' + d'\} = \{a + b, c + d\}$ , then  $G_{t'}^{s'}(a', b'; c', d') \in \mathcal{G}_r(a, b; c, d)$ , where  $r = s + t$ .*

**Proof.** Since  $G_{t'}^{s'}(a', b'; c', d') \sim G_t^s(a, b; c, d)$ , we have  $a' + b' + c' + d' + s' + t' = |E(G_{t'}^{s'}(a', b'; c', d'))| = |E(G_t^s(a, b; c, d))| = a + b + c + d + s + t$ . But  $a' + b' + c' + d' = a + b + c + d$ . Thus we have  $s' + t' = s + t$ . By Lemma 1 and Eq. (1), we have

$$P(G_t^s(a, b; c, d)) = P(X) + (-1)^{s+t} P(G_0^0(a, b; c, d))$$

and

$$P(G_{t'}^{s'}(a', b'; c', d')) = P(Y) + (-1)^{s'+t'} P(G_0^0(a', b'; c', d')),$$

where  $X$  denotes the graph consisting of three cycles  $C_{a+b}$ ,  $C_{c+d}$  and  $C_{s+t+1}$  with the first two cycles having a common vertex and the last two cycles having a common edge, and  $Y$  denotes the graph consisting of three cycles  $C_{a'+b'}$ ,  $C_{c'+d'}$  and  $C_{s'+t'+1}$  where the first two cycles have a common vertex and the last two cycles have a common edge.

Now, since  $\{a + b, c + d\} = \{a' + b', c' + d'\}$  and  $s + t = s' + t'$ , by Theorem 2 we have  $P(X) = P(Y)$ . Thus, we must have  $P(G_0^0(a, b; c, d)) = P(G_0^0(a', b'; c', d'))$ . Therefore, it follows from Theorem 5 that  $\{a', b', c', d'\} = \{a, b, c, d\}$ . It is now not difficult to see that this equation, together with  $\{a' + b', c' + d'\} = \{a + b, c + d\}$ , imply that  $G_{t'}^{s'}(a', b'; c', d') \in \mathcal{G}_r(a, b; c, d)$ .  $\square$

### 3. Computing $P(G_t^s(a, b; c, d))$

We calculate the chromatic polynomial of  $G_t^s(a, b; c, d)$  in this section. We need this polynomial to prove Theorems 6 and 7. Recall that  $P(G_t^s(a, b; c, d)) = P(G_r^0(a, b; c, d))$  by Lemma 1. First, using the formula  $P(G) = P(G+e) + P(G \cdot e)$  obtained from Theorem 1, we compute the chromatic polynomial of  $G_0^0(a, b; c, d)$ . Let  $x = 1 - \lambda$ ,  $p = a + b$

and  $q = c + d$ .

$$\begin{aligned} P(G_0^0(a, b; c, d)) &= \frac{1}{x^3(x-1)^3} P(C_{a+1})P(C_{b+1})P(C_{c+1})P(C_{d+1}) \\ &\quad - \frac{1}{(x-1)^3} P(C_a)P(C_b)P(C_c)P(C_d) \\ &= \frac{(-1)^{p+q-1}x}{(x-1)^2} \left[ (1+x+x^2) - (x+1)(x^a+x^b+x^c+x^d) \right. \\ &\quad \left. + (x^{a+b}+x^{a+c}+x^{a+d}+x^{b+c}+x^{b+d}+x^{c+d}) - x^{p+q-1} \right]. \end{aligned}$$

We now compute the chromatic polynomial of  $G_t^s(a, b; c, d)$ . Recall that  $r = s + t$ . Let the order of  $G_t^s(a, b; c, d)$  be  $n$ . Therefore,  $n = a + b + c + d + r - 2 = p + q + r - 2$ . Then by using Eq. (1), we have

$$\begin{aligned} P(G_t^s(a, b; c, d)) &= \frac{-1}{x(x-1)^2} P(C_p)P(C_q)P(C_{r+1}) + (-1)^r \cdot P(G_0^0(a, b; c, d)) \\ &= \frac{(-1)^{p+q+r-2}x^2}{(x-1)^2} \left[ x^{p+q+r-2} - x^{p+q-2} - x^{p+r-1} \right. \\ &\quad \left. - x^{q+r-1} + x^{p-1} + x^{q-1} + x^r - 1 \right] + (-1)^r \cdot \frac{(-1)^{p+q-1}x}{(x-1)^2} \\ &\quad \times \left[ (1+x+x^2) - (x+1)(x^a+x^b+x^c+x^d) \right. \\ &\quad \left. + (x^{a+b}+x^{a+c}+x^{a+d}+x^{b+c}+x^{b+d}+x^{c+d}) - x^{p+q-1} \right] \\ &= \frac{(-1)^n x}{(x-1)^2} \cdot Q(G_t^s(a, b; c, d)), \end{aligned}$$

where

$$\begin{aligned} Q(G_t^s(a, b; c, d)) &= (x^{n+1} - x^{a+b+r} - x^{c+d+r} + x^{r+1} - x) \\ &\quad - (1+x+x^2) + (x+1)(x^a+x^b+x^c+x^d) \\ &\quad - (x^{a+c}+x^{a+d}+x^{b+c}+x^{b+d}). \end{aligned}$$

#### 4. Chromatic equivalence classes

Suppose that  $H$  is a graph such that  $P(H) = P(G_t^s(a, b; c, d))$ . Then by Theorem 4, we know that  $H$  is also a generalized polygon tree with three interior regions and intercourse number two. Thus  $H = G_{t'}^{s'}(a', b'; c', d')$ , where  $r' = s' + t' \geq 1$  and  $\min\{a', b', c', d'\} = d' \geq 2$ . The question now is whether or not the graph  $G_{t'}^{s'}(a', b'; c', d')$  is in the family  $\mathcal{C}_r(a, b; c, d)$ . In other words, is  $\mathcal{C}_r(a, b; c, d) = \langle G_t^s(a, b; c, d) \rangle$ ? In this section, we present several sufficient conditions so that  $\mathcal{C}_r(a, b; c, d) = \langle G_t^s(a, b; c, d) \rangle$  holds.

Our first sufficiency result is the following theorem.

**Theorem 6.** *If  $\min\{a, b, c, d\} \geq r + 3 \geq 4$ , then  $\mathcal{C}_r(a, b; c, d)$  is a chromatic equivalence class.*

**Proof.** Let  $G = G_i^s(a, b; c, d) \in \mathcal{C}_r(a, b; c, d)$ . Suppose that  $H$  is a graph such that  $H \sim G_i^s(a, b; c, d)$ . Then by Theorem 4,  $H$  is also a generalized polygon tree with three interior regions and  $c(H) = c(G) = 2$ , that is,  $H = G_{t'}^{s'}(a', b'; c', d')$  where  $a', b', c', d' \geq 2$  and  $s' + t' = r' \geq 1$ . By Lemma 2, we need only show that  $\{a' + b', c' + d'\} = \{a + b, c + d\}$ . Now we solve the equation  $Q(G) = Q(H)$ . After cancelling the terms  $x^{n+1}$ ,  $-x$  and  $-(1 + x + x^2)$ , we get  $Q_1(G) = Q_1(H)$ , where

$$Q_1(G) = x^{r+1} + (x+1)(x^a + x^b + x^c + x^d) - x^{r+a+b} - x^{r+c+d} \\ - x^{a+c} - x^{a+d} - x^{b+c} - x^{b+d},$$

$$Q_1(H) = x^{r'+1} + (x+1)(x^{a'} + x^{b'} + x^{c'} + x^{d'}) - x^{r'+a'+b'} \\ - x^{r'+c'+d'} - x^{a'+c'} - x^{a'+d'} - x^{b'+c'} - x^{b'+d'},$$

$$r + a + b + c + d = r' + a' + b' + c' + d', a \geq b, c \geq d, b \geq d,$$

$$a' \geq b' \geq d', \text{ and } c' \geq d'.$$

**Claim.**  $\min\{r+1, a, b, c, d\} = \min\{r'+1, a', b', c', d'\}$ .

To show this claim, let  $\min\{r+1, a, b, c, d\} = \alpha$  and  $\min\{r'+1, a', b', c', d'\} = \beta$ . Now, note that  $x^\alpha$  in  $Q_1(G)$  cannot be cancelled by any negative term of  $Q_1(G)$ , and similarly  $x^\beta$  in  $Q_1(H)$  also cannot be cancelled by any negative term of  $Q_1(H)$ . If  $\alpha > \beta$ , then  $x^\beta$  appears in  $Q_1(H)$  but not in  $Q_1(G)$ , which is impossible. Similarly, if  $\alpha < \beta$ , then we have  $x^\alpha$  in  $Q_1(G)$  but not in  $Q_1(H)$ , and this is also impossible. Thus, we must have  $\alpha = \beta$  as claimed.

Since  $\min\{a, b, c, d\} \geq r + 3$ , we must have  $\min\{r+1, a, b, c, d\} = r + 1$ . Also note that  $\min\{r'+1, a', b', c', d'\}$  equals either  $r' + 1$  or  $d'$ . By the claim above, we have either  $r = r'$  or  $r + 1 = d'$ . We consider two cases:  $r = r'$  and  $r + 1 = d'$ .

Case 1:  $r = r'$ .

In this case, we have  $Q_2(G) = Q_2(H)$ , where

$$Q_2(G) = (x+1)(x^a + x^b + x^c + x^d) - x^{r+a+b} - x^{r+c+d} \\ - x^{a+c} - x^{a+d} - x^{b+c} - x^{b+d},$$

$$Q_2(H) = (x+1)(x^{a'} + x^{b'} + x^{c'} + x^{d'}) - x^{r+a'+b'} \\ - x^{r+c'+d'} - x^{a'+c'} - x^{a'+d'} - x^{b'+c'} - x^{b'+d'},$$

$$a + b + c + d = a' + b' + c' + d', a \geq b \geq d, c \geq d, r + 3 \leq d,$$

$$a' \geq b' \geq d', \text{ and } c' \geq d'.$$

Note that no cancellation is possible between the lowest power term  $x^d$  (respectively,  $x^{d'}$ ) and any of the negative terms in  $Q_2(G)$  (respectively,  $Q_2(H)$ ). Thus, we must have  $d = d'$ . Therefore,  $Q_3(G) = Q_3(H)$ , where

$$\begin{aligned} Q_3(G) &= (x+1)(x^a + x^b + x^c) - x^{r+a+b} - x^{r+c+d} \\ &\quad - x^{a+c} - x^{a+d} - x^{b+c} - x^{b+d}, \\ Q_3(H) &= (x+1)(x^{a'} + x^{b'} + x^{c'}) - x^{r+a'+b'} \\ &\quad - x^{r+c'+d} - x^{a'+c'} - x^{a'+d} - x^{b'+c'} - x^{b'+d}, \\ a+b+c &= a'+b'+c', a \geq b \geq d, c \geq d, r+3 \leq d, \\ a' &\geq b' \geq d, \text{ and } c' \geq d. \end{aligned}$$

Again, it is easy to see that  $\min\{a, b, c\} = \min\{a', b', c'\}$ . We now consider two subcases:  $b \geq c$  and  $b < c$ .

*Subcase 1.1:  $b \geq c$ .*

In this subcase,  $\min\{a, b, c\} = c$ . Thus if  $b' \geq c'$ , then  $c = c'$  and we have  $a' + b' = a + b$  and  $c' + d' = c + d$  as desired. On the other hand, if  $b' < c'$ , then we have  $c = b'$ . Since the girth of  $G$  and  $H$  are equal to  $c + d$ , we must have  $a' = d$  or  $c' = c$ . If  $c' = c$ , then we are done. If  $a' = d$ , then we have  $a' + b' = d + c$  and  $c' + d' = b + a$ . Therefore,  $\{a' + b', c' + d'\} = \{a + b, c + d\}$  as desired.

*Subcase 1.2:  $b < c$ .*

In this subcase, we have  $\min\{a, b, c\} = b$ . Again we consider two possibilities. The first possibility is that  $b' \geq c'$ . Here we have  $b = c'$  and we get  $Q_4(G) = Q_4(H)$ , where

$$\begin{aligned} Q_4(G) &= (x+1)(x^a + x^c) - x^{r+a+b} - x^{r+c+d} - x^{a+c} \\ &\quad - x^{a+d} - x^{b+c} - x^{b+d}, \\ Q_4(H) &= (x+1)(x^{a'} + x^{b'}) - x^{r+a'+b'} - x^{r+b+d} \\ &\quad - x^{a'+b} - x^{a'+d} - x^{b'+b} - x^{b'+d}, \\ a+c &= a' + b', a \geq b \geq d, c \geq d, b < c, r+3 \leq d, \\ a' &\geq b' \geq d, \text{ and } c' \geq d. \end{aligned}$$

Note that the girth of  $H$  is  $c' + d' = b + d$ . Since  $b < c$  and the girths of  $G$  and  $H$  are equal to  $b + d$ , we must have  $a + b = b + d$ , so that  $a = d$ . Thus, we have  $\{a' + b', c' + d'\} = \{a + c, b + d\} = \{d + c, a + b\}$  as required.

Now we consider the second possibility, that  $b' < c'$ . Here we have  $b = b'$  and we get  $Q_5(G) = Q_5(H)$ , where

$$\begin{aligned} Q_5(G) &= (x+1)(x^a + x^c) - x^{r+a+b} - x^{r+c+d} - x^{a+d} - x^{b+c}, \\ Q_5(H) &= (x+1)(x^{a'} + x^{c'}) - x^{r+a'+b} - x^{r+c'+d} - x^{a'+d} - x^{b+c'}, \\ a+c &= a' + c', a \geq b \geq d, c \geq d, b < c, r+3 \leq d, \\ a' &\geq b, c' \geq d, \text{ and } c' > b. \end{aligned}$$

By comparing the least power positive terms of  $Q_5(G)$  and  $Q_5(H)$ , we have  $\min\{a, c\} = \min\{a', c'\}$ . First, we suppose  $\min\{a, c\} = c$ . If  $a' \geq c'$ , then we have  $a' = a$  and  $c' = c$ , and we are done. If  $a' < c'$ , then  $a' = c$  and  $c' = a$ . In this case, we get  $Q_6(G) = Q_6(H)$ , where

$$\begin{aligned} Q_6(G) &= -x^{r+a+b} - x^{r+c+d} - x^{a+d} - x^{b+c}, \\ Q_6(H) &= -x^{r+c+b} - x^{r+a+d} - x^{c+d} - x^{b+a}, \\ a &\geq c > b \geq d. \end{aligned}$$

Note that the lowest power term in  $Q_6(H)$  is  $-x^{c+d}$ , which is obviously not one of the first two terms of  $Q_6(G)$ . Thus, the lowest power term in  $Q_6(G)$  is either  $-x^{a+d}$  or  $-x^{b+c}$ . Therefore, by comparing the lowest power terms of  $Q_6(H)$  and  $Q_6(G)$ , we have either  $c+d = a+d$  or  $c+d = b+c$ , that is, either  $a=c$  or  $b=d$ . In both cases, we have  $\{a' + b', c' + d'\} = \{a+b, c+d\}$  as required.

We now let  $\min\{a, c\} = a$ . If  $a' < c'$ , then we have  $a' = a$  and  $c' = c$ , and we are done. If  $a' \geq c'$ , then  $a' = c$  and  $c' = a$ . In this case, we also have either  $a=c$  or  $b=d$ , and both cases give us  $\{a' + b', c' + d'\} = \{a+b, c+d\}$  as desired.

Case 2:  $r+1 = d'$ .

In this case, we get  $Q_7(G) = Q_7(H)$ , where

$$\begin{aligned} Q_7(G) &= (x+1)(x^a + x^b + x^c + x^d) - x^{r+a+b} - x^{r+c+d} \\ &\quad - x^{a+c} - x^{a+d} - x^{b+c} - x^{b+d}, \\ Q_7(H) &= x^{r'+1} + (x+1)(x^{a'} + x^{b'} + x^{c'}) + x^{r+2} - x^{r'+a'+b'} \\ &\quad - x^{r'+c'+r+1} - x^{a'+c'} - x^{a'+r+1} - x^{b'+c'} - x^{b'+r+1}, \\ a+b+c+d &= r' + a' + b' + c' + 1, a \geq b \geq d, c \geq d, \\ a' \geq b' &\geq r+1, \text{ and } c' \geq r+1. \end{aligned}$$

We note that the positive term  $x^{r+2}$  of  $Q_7(H)$  cannot be cancelled by any negative term of  $Q_7(H)$ . Since  $\min\{a, b, c, d\} \geq r+3$ , the equation  $Q_7(G) = Q_7(H)$  has no solution.

The proof of the theorem is now complete.  $\square$

**Remark.** The condition that  $\min\{a, b, c, d\} \geq 4$  in Theorem 6 above is best possible in the sense that if  $\min\{a, b, c, d\} = 3$ , then  $\mathcal{C}_r(a, b; c, d)$  is not a chromatic equivalence class. For example,  $G_3^0(4, 3; 5, 2) \notin \mathcal{C}_1(6, 3; 4, 3)$  but  $G_3^0(4, 3; 5, 2) \sim G_1^0(6, 3; 4, 3)$ .

**Corollary** (Peng [5]). *If  $\min\{a, b, c, d\} \geq 4$ , then  $G_1^0(a, b; c, d)$  is a chromatically unique graph.*

In Theorem 6,  $r$  is smaller than  $a, b, c$  and  $d$ . For large  $r$ , we have the following result.



**Theorem 7.** If  $r \geq a \geq b+3 \geq c+6 \geq d+9$ , then  $\mathcal{C}_r(a, b; c, d)$  is a chromatic equivalence class.

**Proof.** Suppose that  $H \sim G_t^s(a, b; c, d) = G$ . Then by Theorem 4,  $H = G_{t'}^{s'}(a', b'; c', d')$ , where  $a', b', c', d' \geq 2$  and  $s' + t' = r' \geq 1$ . We now solve the equation  $Q(G) = Q(H)$ , and after cancellation of equal terms, we get  $Q_1(G) = Q_1(H)$ , where

$$\begin{aligned} Q_1(G) &= x^{r+1} + (x+1)(x^a + x^b + x^c + x^d) - x^{r+a+b} - x^{r+c+d} \\ &\quad - x^{a+c} - x^{a+d} - x^{b+c} - x^{b+d}, \\ Q_1(H) &= x^{r'+1} + (x+1)(x^{a'} + x^{b'} + x^{c'} + x^{d'}) - x^{r'+a'+b'} \\ &\quad - x^{r'+c'+d'} - x^{a'+c'} - x^{a'+d'} - x^{b'+c'} - x^{b'+d'}, \\ r+a+b+c+d &= r'+a'+b'+c'+d', r \geq a \geq b+3 \geq c+6 \geq d+9, \\ a' &\geq b' \geq d', \text{ and } c' \geq d'. \end{aligned}$$

Note that  $\min\{r+1, a, b, c, d\} = d$  and  $\min\{r'+1, a', b', c', d'\}$  is equal to either  $r'+1$  or  $d'$ . Since  $\min\{r+1, a, b, c, d\} = \min\{r'+1, a', b', c', d'\}$  as shown in the claim of the proof of Theorem 6, we have either  $d = d'$  or  $d = r'+1$ . We consider two cases:  $d = d'$  and  $d = r'+1$ .

Case 1:  $d = d'$ .

In this case, we get  $Q_2(G) = Q_2(H)$ , where

$$\begin{aligned} Q_2(G) &= x^{r+1} + (x+1)(x^a + x^b + x^c) - x^{r+a+b} - x^{r+c+d} \\ &\quad - x^{a+c} - x^{a+d} - x^{b+c} - x^{b+d}, \\ Q_2(H) &= x^{r'+1} + (x+1)(x^{a'} + x^{b'} + x^{c'}) - x^{r'+a'+b'} \\ &\quad - x^{r'+c'+d} - x^{a'+c'} - x^{a'+d} - x^{b'+c'} - x^{b'+d}, \\ r+a+b+c &= r'+a'+b'+c', r \geq a \geq b+3 \geq c+6 \geq d+9, \\ a' &\geq b' \geq d, \text{ and } c' \geq d. \end{aligned}$$

By comparing the least powers of the positive terms in  $Q_2(G)$  and  $Q_2(H)$ , we have  $c = \min\{a, b, c, r+1\} = \min\{a', b', c', r'+1\}$  which is  $r'+1$ ,  $b'$ , or  $c'$ . We now consider two subcases:  $b' \geq c'$  and  $b' < c'$ .

Subcase 1.1:  $b' \geq c'$ .

In this case, we have either  $c = r'+1$  or  $c = c'$ . If the former holds, then after cancelling  $x^c$  and  $x^{r'+1}$ , the lowest power positive term of  $Q_2(G)$  is  $x^{c+1}$ . Since this term cannot be cancelled by any negative term of  $Q_2(G)$ , we must have  $c+1 = c'$ , since  $x^{c'}$  cannot be cancelled by any negative term in  $Q_2(H)$ . Now the positive term  $x^{c'+1}$  in  $Q_2(H)$  cannot be cancelled by any negative term of  $Q_2(H)$  but  $x^{c'+1} = x^{c+2}$  which is not a term in  $Q_2(G)$ . Therefore,  $c = r'+1$  is impossible. If  $c = c'$ , then we

get  $Q_3(G) = Q_3(H)$ , where

$$\begin{aligned} Q_3(G) &= x^{r+1} + (x+1)(x^a + x^b) - x^{r+c+d} \\ &\quad - x^{a+c} - x^{a+d} - x^{b+c} - x^{b+d}, \\ Q_3(H) &= x^{r'+1} + (x+1)(x^{a'} + x^{b'}) - x^{r'+c+d} \\ &\quad - x^{a'+c} - x^{a'+d} - x^{b'+c} - x^{b'+d}, \end{aligned}$$

$$r+a+b=r'+a'+b', r \geq a \geq b+3 \geq c+6 \geq d+9, \text{ and } a' \geq b' \geq d.$$

Now,  $b = \min\{r+1, a, b\} = \min\{r'+1, a', b'\}$ . Thus, we have either  $b = r'+1$  or  $b = b'$ . If the former holds, then after cancelling  $x^b$  in  $Q_3(G)$  and  $x^{r'+1}$  in  $Q_3(H)$ , the positive term  $x^{b+1}$  in  $Q_3(G)$  is the lowest power positive term in  $Q_3(G)$  and it cannot be cancelled by any negative term of  $Q_3(G)$ . Since  $x^{b'}$  or  $x^{b'+1}$  of  $Q_3(H)$  is the lowest power positive term of  $Q_3(H)$  left uncanceled, we must have  $b' = b$  or  $b' = b+1$ . Therefore,  $b'+1 \leq b+2 = r'+3 < r'+c+d$ , so that neither  $x^{b'}$  nor  $x^{b'+1}$  can be cancelled by a negative term of  $Q_3(H)$ . Hence  $b' = b+1$ . Now  $x^{b'+1} = x^{b+2}$  which is in  $Q_3(H)$  and it cannot be cancelled by any negative term of  $Q_3(H)$ , but  $x^{b+2}$  is not in  $Q_3(G)$ . Thus  $b = r'+1$  is impossible. Therefore, we must have  $b = b'$  and we get  $Q_4(G) = Q_4(H)$ , where

$$\begin{aligned} Q_4(G) &= x^{r+1} + (x+1)x^a - x^{r+c+d} - x^{a+c} - x^{a+d}, \\ Q_4(H) &= x^{r'+1} + (x+1)x^{a'} - x^{r'+c+d} - x^{a'+c} - x^{a'+d}, \\ r+a &= r'+a', r \geq a \geq b+3 \geq c+6 \geq d+9, \text{ and } a' \geq b. \end{aligned}$$

Suppose that  $a \neq a'$ . Thus,  $r \neq r'$ . Since  $x^a$  and  $x^{a+1}$  in  $Q_4(G)$  cannot be cancelled by any negative term of  $Q_4(G)$ , we must have either  $a = r'+1$  and  $a+1 = a'$ , or  $a = a'+1$  and  $a+1 = r'+1$ . The latter case gives us  $a'+r' = 2a-1 = a+r$ . Thus, we have  $r = a-1$ , contradicting our assumption  $r \geq a$ . The former case yields  $r = a$  and we get  $Q_5(G) = Q_5(H)$ , where

$$\begin{aligned} Q_5(G) &= x^{r+1} - x^{r+c+d} - x^{r+c} - x^{r+d}, \\ Q_5(H) &= x^{r+2} - x^{r+c+d-1} - x^{r+c+1} - x^{r+d+1}. \end{aligned}$$

It is now clear that  $Q_5(G) \neq Q_5(H)$ . Thus  $a = a'$  and the solution of the equation  $Q(G) = Q(H)$  is  $a' = a$ ,  $b' = b$ ,  $c' = c$ ,  $d' = d$  and  $r' = r$  as desired.

*Subcase 1.2:  $b' < c'$ .*

In this subcase, we have either  $c = r'+1$  or  $c = b'$ . As in subcase 1.1 above,  $c = r'+1$  is impossible. If  $c = b'$ , then since the girths of  $G$  and  $H$  equal  $c+d$ , we must have  $c' = c$  or  $a' = d$ . If  $c' = c$ , then  $c' = c = b' < c'$  which is absurd. If  $a' = d$ , then  $d = a' \geq b' = c$ , contradicting our assumption that  $c \geq d+3$ .

*Case 2:  $d = r'+1$ .*

In this case, we get  $Q_6(G) = Q_6(H)$ , where

$$Q_6(G) = x^{r+1} + (x+1)(x^a + x^b + x^c) + x^{d+1} - x^{r+a+b} \\ - x^{r+c+d} - x^{a+c} - x^{a+d} - x^{b+c} - x^{b+d},$$

$$Q_6(H) = (x+1)(x^{a'} + x^{b'} + x^{c'}) + x^{d'} + x^{d'+1} - x^{r'+a'+b'} \\ - x^{r'+c'+d'} - x^{a'+c'} - x^{a'+d'} - x^{b'+c'} - x^{b'+d'},$$

$$r+a+b+c+1 = a'+b'+c'+d', r \geq a \geq b+3 \geq c+6 \geq d+9,$$

$$a' \geq b' \geq d', \text{ and } c' \geq d'.$$

Now, we have  $\min\{r+1, a, b, c, d+1\} = \min\{a', b', c', d'\}$ . Therefore,  $d+1 = d'$  and  $x^{d+1}$  in  $Q_6(G)$  cancels  $x^{d'}$  in  $Q_6(H)$ . The positive term  $x^{d'+1} = x^{d+2}$  in  $Q_6(H)$  cannot be cancelled by any negative term of  $Q_6(H)$  but  $x^{d+2}$  is not a term in  $Q_6(G)$ . Thus, the equation  $Q(G) = Q(H)$  has no solution.  $\square$

By a method similar to that used in proving Theorems 6 and 7, we have the following three results. (The detail proofs of these theorems will be made available to a reader upon request from the first author.)

**Theorem 8.** *If  $r-c \geq a \geq c \geq b+3 \geq 5$ , then  $\mathcal{C}_r(a, b; c, b)$  is a chromatic equivalence class.*

**Theorem 9.** *If  $r-c \geq a \geq b \geq c+3 \geq 5$ , then  $\mathcal{C}_r(a, b; c, c)$  is a chromatic equivalence class.*

**Theorem 10.** *If  $r \neq b+1$  and  $a \geq b+3 \geq 5$ , then  $\mathcal{C}_r(a, b; a, b)$  is a chromatic equivalence class.*

We conclude this paper with the following conjecture which is true for  $r=1$  (see [5]).

**Conjecture.** The family of graphs  $\mathcal{C}_r(a, b; c, d)$  is a chromatic equivalence class whenever  $a, b, c, d$  are each at least four.

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